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Generalized $N = 2$ Supersymmetric Toda Field Theories

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Abstract

In this paper we introduce a class of generalized supersymmetric Toda field theories. The theories are labeled by a continuous parameter and have $N = 2$ supersymmetry. They include previously known $N = 2$ Toda theories as special cases. Using the WZNW→Toda reduction approach we obtain a closed expression for the bracket of the associated $\mathcal{W}$ algebras. We also derive an expression for the generators of the $\mathcal{W}$ algebra in a free superfield realization.

1 Introduction

In a recent paper [1] we considered a certain class of generalized (classical) conformal Toda field theories, proposed earlier by Brink and Vasiliev [2]. It is natural to extend this construction to supersymmetric Toda theories. Supersymmetric conformal Toda theories based on finite-dimensional simple Lie superalgebras have been described previously in the literature [3, 4, 5]. In this paper we will describe a class of infinite-dimensional generalized supersymmetric Toda theories. The theories are labeled by a continuous parameter, such that when this parameter takes certain discrete values the model reduces to a finite-dimensional supersymmetric Toda theory. The generalized models will turn out to have $N = 2$ supersymmetry. Through a reduction procedure, which will be described in more detail later, it is also possible to obtain generalized Toda theories with $N = 1$ supersymmetry. We will only discuss the classical models, and leave the quantization for a future publication. The paper is organized as follows. In the next section we discuss the general Lie superalgebra on which our construction is based, and derive some results which will be used in later sections. In section 3 we define and describe the generalized models, using the WZNW reduction approach to supersymmetric Toda theories, developed earlier for the finite-dimensional cases [6]. Associated with the generalized Toda models we find a set of generalized $\mathcal{W}$ algebras. As in the bosonic case [1] it is possible to obtain a closed expression for the bracket used to calculate the $\mathcal{W}$ algebra associated with the generalized Toda theories. The calculation of this (Dirac) bracket is presented in section 4. Finally, in the last section we discuss a free superfield realization of the general $\mathcal{W}$ algebra, and give a closed expression for the $\mathcal{W}$ algebra generators in terms of these free fields, i.e. a Miura transformation. We apply the method to a simple case as an example. The (finite-dimensional) conformal supersymmetric Toda theories are described in detail in [3]. For more about WZNW→Toda reduction the review [7] contains useful information.

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2 The General Lie Superalgebra

The general Lie superalgebra upon which our construction is based has been described before [8, 9, 2, 1]. We will therefore concentrate on the points which are most relevant for the present paper. The general Lie superalgebra is defined as the algebra whose elements are monomials in the three operators K and $a^{\pm}$ satisfying

$$\begin{aligned} [a^-, a^+] &= 1 + 2\nu K \,, \\ \{K, a^{\pm}\} &= 0 \;\;, \quad K^2 = 1 \,. \end{aligned} \tag{2.1}$$

A general element can therefore be written as

$$B = \sum_{A=0}^{1} \sum_{n=0}^{\infty} \frac{1}{n!} b^A_{\alpha_1 \ldots \alpha_n} K^A \, a^{\alpha_1} \ldots a^{\alpha_n} \,. \tag{2.2}$$

We can choose $b^A_{\alpha_1 \ldots \alpha_n}$ to be totally symmetric in the lower indices, so the basis elements can be chosen to be the Weyl ordered products

$$E_{nm} = \Big((a^+)^n (a^-)^m\Big)_{\text{Weyl}} = \frac{1}{(n+m)!}\Big((a^+)^n (a^-)^m + ((n+m)! - 1))\,\text{permutations}\Big) \,, \tag{2.3}$$

multiplied by an appropriate function of the Klein operator K. Because of the Klein operator there are two independent basis elements for fixed n and m. The (general) algebra possesses a natural $\mathbf{Z}_2$ gradation. The grade $[A]$ of an element T^A in the algebra is defined to be equal to 0 or 1, for monomials with even or odd powers n in (2.2), respectively. The graded bracket satisfy the usual property

$$[T^A, T^B\} = -(-1)^{[A][B]} [T^B, T^A\} \,. \tag{2.4}$$

The operators $T^{\pm} = \frac{1}{2}(a^+)^2$ and $T^0 = \frac{1}{4}\{a^-, a^{\pm}\}$ span an sl_2 subalgebra of the general Lie superalgebra. An important point for this paper is that the general algebra has an $osp(1|2)$ subalgebra

$$\begin{aligned} \{a^{\pm}, a^{\pm}\} = 4T^{\pm} \quad &, \quad \{a^+, a^-\} = 4T^0 \,, \\ [T^0, a^{\pm}] = \pm\frac{1}{2} a^{\pm} \quad &, \quad [T^{\pm}, a^{\mp}] = \mp a^{\pm} \,, \\ [T^0, T^{\pm}] = \pm T^{\pm} \quad &, \quad [T^-, T^+] = 2T^0 \,, \end{aligned} \tag{2.5}$$

which can be extended to an $sl(2|1) \cong osp(2|2)$ subalgebra

$$\begin{aligned} \{\bar{c}^+, \bar{c}^-\} = 2h \quad &, \quad \{c^+, c^-\} = 2\bar{h} \,, \\ [h, c^{\pm}] = \pm c^{\pm} \quad &, \quad [\bar{h}, \bar{c}^{\pm}] = \pm \bar{c}^{\pm} \,, \\ \{c^{\pm}, \bar{c}^{\pm}\} = 2T^{\pm} \quad &, \quad [T^{\pm}, c^{\mp}] = \mp \bar{c}^{\pm} \,, \\ [T^{\pm}, \bar{c}^{\mp}] = \mp c^{\pm} \quad &, \quad [h, T^{\pm}] = \pm T^{\pm} \,, \\ [\bar{h}, T^{\pm}] = \pm T^{\pm} \quad &, \quad [T^-, T^+] = h + \bar{h} \,. \end{aligned} \tag{2.6}$$

Here

$$
\begin{aligned}
c^{\pm} = \mathrm{P}_{\pm} a^{\pm} \quad &, \quad \bar{c}^{\mp} = \mathrm{P}_{\pm} a^{\mp}\,, \\
h = T^0 + \frac{K}{4}(1+2\nu K) \quad &, \quad \bar{h} = T^0 - \frac{K}{4}(1+2\nu K)\,,
\end{aligned}
\tag{2.7}
$$

and $\mathrm{P}_{\pm} = \frac{1\pm K}{2}$. The algebra has a unique invariant supertrace operation [8, 9], defined as

$$
str(B) = b^0 - 2\nu b^1\,, \tag{2.8}
$$

where B is of the form (2.2), with the property

$$
str(T^A T^B) = (-1)^{[A]} str(T^B T^A) = (-1)^B str(T^B T^A)\,. \tag{2.9}
$$

The properties of the bilinear form $\langle\cdot,\cdot\rangle$ defined as $\langle T^A, T^B\rangle = str(T^A T^B)$, can thus be collected together as

$$
\begin{array}{rcll}
\langle T^A, T^B\rangle & = & 0, \text{ if } [A] \neq [B] & \text{(consistency)} \\
\langle T^A, T^B\rangle & = & (-1)^{[A][B]}\langle T^B, T^A\rangle & \text{(supersymmetry)} \\
\langle [T^A, T^B\} T^C\rangle & = & \langle T^A [T^B, T^C\}\rangle & \text{(invariance)}\,.
\end{array}
\tag{2.10}
$$

We also have the formula

$$
str(K^A E_{nm} E_{rs}) = \delta_{mr}\delta_{ns}(-1)^n n! m! \beta^A(n+m)\,, \tag{2.11}
$$

with $\beta^A(0) = [\delta_{A,0} - 2\nu\delta_{A,1}]$ and

$$
\beta^A(n) = 2^{-n}\left[\delta_{A,0} - \frac{1}{2}(1+(-1)^n)\frac{2\nu}{n+1}\delta_{A,1}\right]\prod_{l=0}^{[(n-1)/2]}(1 - \frac{4\nu^2}{(2l+1)^2})\,. \tag{2.12}
$$

The algebra under discussion can be shown to be isomorphic to the universal enveloping algebra of $osp(1|2)$, $U(osp(1|2))$, divided by a certain ideal [10]. When $d = n$, where $d = \frac{2\nu+1}{2}$ and n is a positive integer, the bilinear form (pseudo scalar product) $\langle\cdot,\cdot\rangle$ becomes degenerate i.e. "null-states" appear, and the algebra becomes, after dividing out the ideal spanned by the null-states, isomorphic to $gl(n|n-1)$ when considered as a Lie superalgebra. From now on we will assume that the ideal 1 ($= E_{00}$) has been factored out from the general Lie superalgebra; this means that we get $sl(n|n-1)$, when $d = n$ (after the ideal spanned by the null-states has been divided out). From the general algebra it is also possible to obtain the algebras $B(n,n) \cong osp(2n-1|2n)$ and $B(n,n-1) \cong osp(2n+1|2n)$ through the following procedure [9]. First we introduce the anti-automorphism[2] $\rho(a^{\pm}) = ia^{\pm}$ and $\rho(K) = K$. From ρ we can obtain an automorphism τ through $\tau(T^A) = -i^{[A]}\rho(T^A)$. Then by extracting the subset of elements satisfying $\tau(T^A) = T^A$ we get (after the appropriate ideal has been divided out) $B(n,n)$ when $d = 2n$ and $B(n,n-1)$ when $d = 2n+1$. It is not possible to obtain the other simple Lie superalgebras through this construction.

[2] Recall that an anti-automorphism ρ satisfies $\rho(UV) = \rho(V)\rho(U)$, whereas an automorphism τ satisfies $\tau(UV) = \tau(U)\tau(V)$.

It will turn out to be convenient to use the following basis of the algebra. For $n+m=$ even, we define

$$
\begin{aligned}
E^1_{nm} &= E_{nm}\,, \\
E^2_{nm} &= \frac{m+1}{2}(K+\frac{2\nu}{m+n+1})E_{nm}\,,
\end{aligned}
\tag{2.13}
$$

and for $n+m=$ odd

$$
\begin{aligned}
E^1_{nm} &= E_{nm}\,, \\
E^2_{nm} &= KE_{nm}\,.
\end{aligned}
\tag{2.14}
$$

From (2.11) it follows that the basis elements satisfy

$$
\begin{aligned}
str(E^1_{nm}E^1_{mn}) &= -str(E^2_{n-1,m-1}E^2_{m-1,n-1}) \quad (n+m=\text{even})\,, \\
str(E^1_{nm}E^1_{mn}) &= -str(E^2_{n,m}E^2_{m,n}) \qquad\qquad\;\; (n+m=\text{odd})\,, \\
str(E^1_{nm}E^2_{pr}) &= 0\,.
\end{aligned}
\tag{2.15}
$$

We will call those elements which are annihilated by ad_{a^+} (ad_{a^-}) highest (lowest) weight elements. We also define the following operators $T=ad_{T^+}=[T^+,\cdot\}$, and $A=ad_{a^+}=[a^+,\cdot\}$. The action of T on the basis elements is

$$
[T^+,E_{n,m}\}=-mE_{n+1,m-1} \quad , \quad [T^+,E_{n,0}\}=0\,. \tag{2.16}
$$

In later sections we will need the action of A on the basis elements. We have, when $n+m$ is even

$$
\begin{aligned}
[a^+,E^1_{n,m}] &= -mE^1_{n,m-1}\,, \\
[a^+,E^2_{n,m}] &= -(m+1)E^2_{n+1,m}(1-\delta_{m,0})\,,
\end{aligned}
\tag{2.17}
$$

and when $n+m$ is odd

$$
\begin{aligned}
\{a^+,E^1_{n,m}\} &= 2E^1_{n+1,m}\,, \\
\{a^+,E^2_{n,m}\} &= 2E^2_{n,m-1}(1-\delta_{m,0})\,.
\end{aligned}
\tag{2.18}
$$

These relations are easily derived e.g. by starting from $E^1_{0,2n}$ and $E^2_{0,2n+1}$, i.e. lowest weight elements, and using the identity $AT^m=T^mA$. The action of T on the basis elements is previously known (cf. (2.16), and the action of A on lowest weight elements is easy to calculate. The properties above mean that A maps the two sectors of the basis elements into themselves (with the exception of highest weight elements, i.e. those elements which are annihilated by ad_{a^+}.) The basis elements can be arranged in the "wedge" shown on the next page. The horizontal lines are irreducible representations of $osp(1|2)$, and A is the step-operator of these representations. This fact makes the orthogonality of the basis elements (2.15) transparent. We will denote the "inverse" of A by a. We have

$$
\begin{aligned}
[a^+,a(x)\} &= x-\Pi_-(x)\,, \\
a([a^+,x\}) &= x-\Pi_+(x)\,,
\end{aligned}
\tag{2.19}
$$

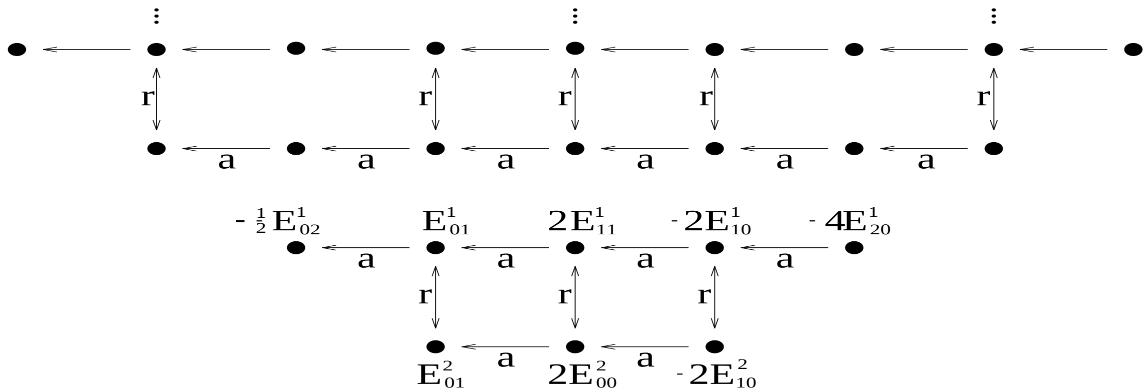

Figure 1: The wedge of $osp(1|2)$ irreducible representations.

where x is an arbitrary element in the algebra, Π_+ is the projection operator which projects onto the space of highest weight elements, and similarly Π_- projects onto lowest weight elements. The action of a on the elements in the algebra, can be inferred from the action of A described above. Furthermore, the following properties are satisfied by A and its "inverse" a

$$\langle UA(V)\rangle = -\langle A(\hat{U})V\rangle \quad , \quad \langle Ua(V)\rangle = \langle a(\hat{U})V\rangle \,, \tag{2.20}$$

where U and V are arbitrary elements of the algebra. The operator $\hat{}$ acts in the following way $\hat{}\ :\ T^A \to (-1)^{[A]}T^A$ on the algebra elements, and is called the grading automorphism. We define the operator r in the following way. When $m+n$ is even

$$\begin{aligned} r(E^i_{nm}) = E^i_{nm}\ \ (n>m) \quad &, \quad r(E^i_{nm}) = -E^i_{nm}\ \ (n<m)\,, \\ r(E^2_{2n,2n}) \ &= \ E^1_{2n+1,2n+1}\,, \\ r(E^1_{2n+1,2n+1}) \ &= \ E^2_{2n,2n}\,, \end{aligned} \tag{2.21}$$

and when $n+m$ is odd

$$\begin{aligned} r(E^1_{nm}) \ &= \ E^2_{nm}\,, \\ r(E^2_{nm}) \ &= \ E^1_{nm}\,. \end{aligned} \tag{2.22}$$

It follows from these definitions that

$$r^2 = 1 \quad , \quad \langle r(U)V\rangle = -\langle Ur(V)\rangle \,, \tag{2.23}$$

and

$$r([U,r(V)\}) + r([r(U),V\}) - [r(U),r(V)\} = [U,V\}\,. \tag{2.24}$$

The first relation is immediate. In the fermionic sector the second relation follows from the fact that in this sector r is equal to multiplication by K, together with the properties of the supertrace; whereas in the bosonic sector it follows from the properties of the supertrace. E.g. for the neutral elements the following combinations

$$h_n = \frac{E^1_{nn} + E^2_{n-1,n-1}}{2} \quad , \quad \bar{h}_n = \frac{E^1_{nn} - E^2_{n-1,n-1}}{2}\,, \tag{2.25}$$

satisfy $r(h_n) = h_n$, $r(\bar{h}_n) = -\bar{h}_n$, and

$$\begin{aligned} str(h_n h_m) = 0 \quad &, \quad str(\bar{h}_n \bar{h}_m) = 0\,, \\ str(h_n \bar{h}_m) \;&\propto\; \delta_{nm}\,. \end{aligned} \tag{2.26}$$

Equation (2.24) follows from the fact that r is equal to K when acting on fermions, together with the fact that the commutator of e.g. two bosonic elements with positive grade is another bosonic element with positive grade. Equation (2.24) is the (classical) modified Yang-Baxter equation. Another interpretation is the following. If we naively set $J = ir$, then (2.23) states that J satisfies the properties of an almost complex structure which is compatible with the metric, whereas (2.24) implies the vanishing of

$$N(U,V) = [U,V\} + J([U,J(V)\}) + J([J(U),V\}) - [J(U),J(V)\}\,, \tag{2.27}$$

which we recognize as the Nijenhuis tensor (field). Hence J is promoted to a complex structure. However, we should remember that we are considering a real algebra and the operator J does not respect the reality condition. Nevertheless, the existence of the operator r is crucial for the appearance of $N = 2$ supersymmetry in the Toda theories described later, which are based on the algebra in this section. However, the supersymmetry is not of the standard form due a sign discrepancy between the algebra of the two sets of supercharges (caused by the fact that r squares to 1 and not to -1). For a discussion of this point we refer to ref. [11].

3 WZNW→Toda Reduction

In this section we will describe the generalized Toda theories using the WZNW→Toda reduction approach. We will start with a brief recapitulation of the procedure for the finite dimensional cases treated earlier [6, 12]. The number of supersymmetries in the resulting Toda theories are related to the existence of certain subalgebras of the underlying Lie superalgebra. These so called principal subalgebras correspond to the different number of supersymmetries which are possible, according to the following rule: An sl_2 principal subalgebra gives rise to $N = 0$, an $osp(1|2)$ principal subalgebra leads to $N = 1$ and finally the existence of an $sl(2|1) \cong osp(2|2)$ principal subalgebra implies $N = 2$ supersymmetry. The generalized Toda theories studied in this paper have $N = 2$ supersymmetry, thus generalizing the finite-dimensional theories based on the $sl(n|n-1)$ Lie superalgebras. We would like to keep the $N = 2$ supersymmetry manifest during the discussion, but unfortunately at the moment the $N = 2$ WZNW model in $N = 2$ superspace is not known. We will therefore (almost exclusively) work in $N = 1$ superspace. See however ref. [13], which discusses certain aspects of a manifestly $N = 2$ treatment. In the $N = 1$ formalism the $N = 2$ symmetry is hidden. The WZNW action in $N = 1$ superspace is[3]

$$\begin{aligned} S_W(g) &= \frac{\kappa}{2}\int d^2z d^2\theta \langle (G^{-1}D_+\hat{G}), (\hat{G}^{-1}D_-G)\rangle \\ &+ \frac{\kappa}{2}\int dt d^2z d^2\theta \langle G^{-1}\partial_t G, (G^{-1}D_+\hat{G}\hat{G}^{-1}D_-G + G^{-1}D_-\hat{G}\hat{G}^{-1}D_+G)\rangle\,. \end{aligned} \tag{3.1}$$

[3] Conventions: $D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\pm\partial_\pm$, $D_\pm^2 = \partial_\pm$.

The superfield $G(z,\theta)$ takes its values in a supergroup, and is of the general form

$$G(z,\theta) = \exp(\Phi_i(z,\theta)B^i + \Psi_i(z,\theta)F^i)\,. \tag{3.2}$$

Here B^i belongs to the set of basis elements of the subalgebra with $\mathbf{Z}_2$ grade 0, and F^i has $\mathbf{Z}_2$ grade 1. Φ_i is a Grassmann even superfield and Ψ_i is Grassmann odd. By identifying the two different gradations, we call a Lie superalgebra valued superfield odd if its total grade is odd, and even if its total grade is even. In this language the WZNW field G is thus even. The grading automorphism $\hat{}$ was defined in section 2. It has the following action on G, $\exp(\Phi_iB^i + \Psi_iF^i) \to \exp(\Phi_iB^i - \Psi_iF^i)$. From now on we set $\kappa = 1$. The WZNW currents obtained from the action (3.1) are

$$J_+ = \hat{G}^{-1}D_+G \quad , \quad J_- = (D_-G)G^{-1}\,. \tag{3.3}$$

These currents can be expanded as $J = \Psi_iB^i + \Phi_iF^i$, i.e. they are odd in the terminology introduced above. The equation of motion can be written as

$$D_-J_+ = D_-(\hat{G}^{-1}D_+G) = 0 \quad \Leftrightarrow \quad D_+J_- = D_+((D_-G)G^{-1}) = 0\,. \tag{3.4}$$

We will denote the Lie superalgebra obtained from the supergroup by $\mathcal{G}$. The algebra is assumed to have an $osp(1|2)$ principal subalgebra of the form (2.5). Algebras of this type include the finite-dimensional algebras listed in [3] as well as the general algebra and its truncations described in the previous section. In the sequel the underlying Lie superalgebra $\mathcal{G}$ is assumed to be the general algebra. In this case however, all statements which involve the concept of a group should be taken to be true on a formal level. The method given below is a formal way to obtain the generalized Toda theories; these can then be defined by their action and associated equations of motion. To obtain the generalized Toda theories the following constraints are imposed on the currents

$$\begin{aligned} \langle E_{\alpha^+}, D_-GG^{-1} - a^-\rangle &\approx 0\,, \\ \langle E_{\alpha^-}, \hat{G}^{-1}D_+G - a^+\rangle &\approx 0\,, \end{aligned} \tag{3.5}$$

for all basis elements E_{α^+} with positive T^0 grade and all basis elements E_{α^-} with negative grade, which means that, on the constraint surface, the currents become

$$J_\pm = a^\pm + j_\pm\,. \tag{3.6}$$

Here $j_\pm$ lies in the Borel subalgebra $\mathcal{G}_0 \oplus \mathcal{G}_\mp$, where $\mathcal{G}_\pm$ are the parts of the algebra with positive and negative T^0 grade respectively, and $\mathcal{G}_0$ is the subset of neutral elements. There is also an alternative lagrangean realization of the reduction in terms of a gauged WZNW model in $N = 1$ superspace, but we will not use it here. Continuing with the general scheme, we make the Gauss decomposition[4] $G = G_+G_0G_-$, where $G_0 = e^\phi$, $\phi \in \mathcal{G}_0$, and $G_\pm$ are the parts of the supergroup which corresponds to the parts $\mathcal{G}_\pm$ of the algebra. The field ϕ can be expanded as $\phi = \sum_n \varphi_nh_n + \bar{\varphi}_n\bar{h}_n$, where φ_n and $\bar{\varphi}_n$ are bosonic superfields. The constraints can be imposed directly into the equation of motion (since they are consistent with the WZNW dynamics). Using the constraint equations

[4]This decomposition is only valid locally. We do not consider global issues [14] in this paper.

(3.5) it is possible to show that, after a similarity transformation with G_-, the equation of motion $D_-(\hat{G}^{-1}D_+G)=0$ becomes

$$D_+D_-\phi = \{a^+, e^{-\phi}a^-e^{\phi}\}\,. \tag{3.7}$$

This equation can be written as a zero curvature condition, i.e.

$$D_-A_+ + D_+A_- + [\mathcal{A}_-, \mathcal{A}_+\} = 0\,, \tag{3.8}$$

where $\mathcal{A}_+ = a^+ + D_+\phi$, and $\mathcal{A}_- = e^{-\phi}a^-e^{\phi}$. The equation of motion (3.7) can be derived from the action[5]

$$S_{\text{Toda}} = \int d^2z d^2\theta str(\frac{1}{2}D_+\phi D_-\phi - a^+e^{-\phi}a^-e^{\phi})\,. \tag{3.9}$$

Since this action is written in $N=1$ superspace it is manifestly $N=1$ invariant, but the above action has actually $N=2$ supersymmetry, this fact can be seen as follows. In the notations of section 2, we write $a^\pm = c^\pm + \bar{c}^\pm$, and $\phi(z^\pm, \theta^\pm, h_n, \bar{h}_n) = \Phi(z^\pm, \theta^\pm) + \bar{\Phi}(z^\pm, \theta^\pm)$. Here $\Phi = \sum_n \varphi_n h_n$ and $\bar{\Phi} = \sum_n \bar{\varphi}_n \bar{h}_n$. We observe that $[c^+, \bar{h}_n] = 0$, and similarly $[\bar{c}^+, h_n] = 0$; together with the property $str(h_n h_m) = 0 = str(\bar{h}_n \bar{h}_m)$, we see that the action can be rewritten as

$$S = \int d^2z d^2\theta str(D_+\Phi D_-\bar{\Phi} - c^+e^{-\Phi}c^-e^{\Phi} - \bar{c}^+e^{-\bar{\Phi}}\bar{c}^-e^{\bar{\Phi}})\,. \tag{3.10}$$

Introducing chiral $N=2$ superfields

$$\Phi_{N=2}(z^\pm, \theta^\pm) = \Phi_{N=1}(\tilde{z}^\pm, \theta^\pm) \quad , \quad \bar{\Phi}_{N=2}(z^\pm, \bar{\theta}^\pm) = \bar{\Phi}_{N=1}(\tilde{\bar{z}}^\pm, \bar{\theta}^\pm)\,, \tag{3.11}$$

where $\tilde{z}^\pm = z^\pm + \theta^\pm\bar{\theta}^\pm$, $\tilde{\bar{z}}^\pm = z^\pm - \theta^\pm\bar{\theta}^\pm$, the action can be written in $N=2$ superspace as[6]

$$S = \int d^2z \left\{ str(d^2\theta d^2\bar{\theta}\Phi\bar{\Phi} - d^2\theta c^+e^{-\Phi}c^-e^{\Phi} - d^2\bar{\theta}\bar{c}^+e^{-\bar{\Phi}}\bar{c}^-e^{\bar{\Phi}}) \right\}\,, \tag{3.12}$$

thereby making the $N=2$ supersymmetry manifest. The equations of motion resulting from the above action are

$$\begin{aligned} D_+D_-\Phi &= \{\bar{c}_+, e^{-\bar{\Phi}}\bar{c}^-e^{\bar{\Phi}}\}\,, \\ \bar{D}_+\bar{D}_-\bar{\Phi} &= \{c_+, e^{-\Phi}c^-e^{\Phi}\}\,. \end{aligned} \tag{3.13}$$

The equations of motion can also be obtained from a zero curvature condition imposed on the $N=2$ Lax connection

$$\begin{aligned} A_+ = D_+\Phi + \bar{c}^+ \quad &, \quad A_- = e^{-\bar{\Phi}}\bar{c}^-e^{\bar{\Phi}}\,, \\ \bar{A}_+ = \bar{D}_-\bar{\Phi} + c^+ \quad &, \quad \bar{A}_- = e^{-\Phi}c^-e^{\Phi}\,. \end{aligned} \tag{3.14}$$

It is also possible to extract the $B(n,n)$ and $B(n,n-1)$ super Toda theories from the above treatment by implementing the automorphism described in section 2. These models have $N=1$ supersymmetry, and are described by the action (3.9), where the fields take values in the appropriate algebra.

[5]This action has been obtained before in ref. [15].

[6]We use $D_+ = \frac{\partial}{\partial\theta^+} + \bar{\theta}^+\partial_+$, $\bar{D}_+ = \frac{\partial}{\partial\bar{\theta}^+} + \theta^+\partial_+$, and similarly for D_-, $\bar{D}_-$.

4 The General $\mathcal{W}$ Algebra

In this section we will derive an expression for the bracket of the supersymmetric $\mathcal{W}$ algebras associated with the generalized supersymmetric Toda theories introduced in the previous section. Considering the great complexity of $\mathcal{W}$ algebras, it is important to be able to make general statements of this type. The formula for the bracket encompasses the entire class of $N=2$ $\mathcal{W}$ algebras labeled by the parameter ν. In particular, by choosing the parameter ν appropriately, the formula applies to the $N=2$ supersymmetric W_n algebras. The general spin content of the finite-dimensional $N=2$ W_n algebras was conjectured in ref. [16, 17], and proven in ref. [18]. Supersymmetric $\mathcal{W}$ algebras also appears in the supersymmetric KdV hierarchies [19, 20].

We now turn to the calculation of the $\mathcal{W}$ algebra bracket. The first class constraints imposed on the currents induces a gauge invariance of the system. The Toda field together with its equation of motion (and $\mathcal{W}$ algebra) is one way to represent the gauge invariant content of the model. It is also possible to describe the properties of the reduced system (in particular the associated $\mathcal{W}$ algebra) in other ways by supplementing the first class constraints

$$\gamma_\alpha = \langle E_\alpha, \hat{G}^{-1} D_+ G - a^+ \rangle \approx 0\,, \tag{4.1}$$

with gauge-fixing conditions. (Here $E_\alpha \in \mathcal{G}_- \oplus \mathcal{G}_0$.) The gauge-fixing conditions together with the first class constraints behave as a set of second class constraints. At the level of the equations of motion, the reduction can be defined on the two (chiral) currents independently, without referring to the WZNW model. Although an $N=(2,2)$ WZNW model in $(2,2)$ superspace is not known, an $N=(2,0)$ affine Kac-Moody algebra (in $(2,0)$ superspace) is known [21]. A hamiltonian reduction in $N=2$ superspace based on the $N=2$ current algebra has been developed in ref. [22]. In this reference the cases of $sl(2|1)$ and $sl(3|2)$ and their associated $\mathcal{W}$ algebras were discussed.

The Poisson bracket between the components of the WZNW current is[7]

$$\{\langle J(X), T^A\rangle, \langle J(Y), T^B\rangle\} = \langle \hat{T}^A [J(X), \hat{T}^B\}\rangle \delta(X-Y) - \langle \hat{T}^A, \hat{T}^B\rangle D_X \delta(X-Y)\,, \tag{4.2}$$

with the definition $\delta(Z_1 - Z_2) = \delta(z_1 - z_2)(\theta_1 - \theta_2)$. The commutator of two superalgebra valued superfields F and G is defined as $[F, G\} = \sum_{A,B} F_A G_B [T^A, T^B\}$. Although our treatment is classical we will occasionally use the perhaps more familiar language of superOPE's, through the formal identification $\delta(Z_1 - Z_2) = \frac{z_{12}}{\theta_{12}}$, where $z_{12} = z_1 - z_2 - \theta_1\theta_2$, and $\theta_{12} = \theta_1 - \theta_2$. Using this rule the Poisson bracket (4.2) can be written

$$\langle J(Z_1), T^A\rangle \langle J(Z_2), T^B\rangle \sim \left[\langle \hat{T}^A [J(Z_1), \hat{T}^B\}\rangle - \langle \hat{T}^A, \hat{T}^B\rangle D_{Z_1}\right] \frac{\theta_{12}}{z_{12}}\,. \tag{4.3}$$

The lowest weight gauge is defined by gauge fixing $j = J - a^+$ to lie in $\ker ad_{a^-}$. In this gauge the current can thus be written as $J = a^+ - W$, where $W \in \ker ad_{a^-}$. The total set of second class constraints can be written as

$$\chi_\alpha = \langle J - a^+, a E_\alpha \rangle \approx 0\,, \tag{4.4}$$

[7] X and Y are $N=1$ (chiral) superspace coordinates, and $J = J_+$. In this section we use the convention $D = \frac{\partial}{\partial\theta} + \theta\partial$, $D^2 = \partial$.

where E_α is not lowest weight, i.e. is not annihilated by a. (a was defined in section 2.) W can be expanded as

$$W(X) = \sum_n W_{n+\frac{1}{2}}(X) \frac{(a^-)^{2n}}{\langle (a^-)^{2n}, (a^+)^{2n} \rangle} + W_n(X) \frac{K(a^-)^{2n-1}}{\langle K(a^-)^{2n-1}, K(a^+)^{2n-1} \rangle}, \tag{4.5}$$

with the understanding that when $d = n$, where n is an integer, the above sum only contains a finite number of terms. Recalling that J is an odd field, we see that the W_n's are Grassmann even whereas the $W_{n+\frac{1}{2}}$'s are Grassmann odd. We proceed as in the bosonic case [1]; the Dirac bracket is defined as

$$\{\cdot,\cdot\}^* = \{\cdot,\cdot\} - \sum_{\alpha\beta} \iint dX dY \{\cdot, \chi_\alpha(X)\} C^{-1}_{\alpha\beta}(X,Y) \{\chi_\beta(Y), \cdot\}. \tag{4.6}$$

The constraint matrix is given by $C_{\alpha\beta}(X,Y) = \{\chi_\alpha(X), \chi_\beta(Y)\}$. Using the expression for the Poisson bracket (4.2) together with (2.20) we get

$$C_{\alpha\beta}(X,Y) = \langle E_\alpha a U_0^{-1}(X) \hat{E}_\beta \rangle \delta(X-Y). \tag{4.7}$$

Here E_α and E_β are not lowest weight. Furthermore $U_0^{-1} = 1 - \mathcal{D}_0 a$, where a is the (almost) inverse of ad_{a^+}, and $\mathcal{D}_0(Z) = D_Z + [W(Z), \cdot\}$. The inverse of $C_{\alpha\beta}$ is, as will be shown below, given by

$$C^{-1}_{\alpha\beta}(X,Y) = -\langle \hat{E}_{-\alpha} \hat{U}_0(X) A \hat{E}_{-\beta} \rangle \delta(X-Y), \tag{4.8}$$

where $A = ad_{a^+} = [a^+, \cdot\}$, and $E_{-\alpha}$ is the dual (in a suitable normalization) of E_α, in the sense $\langle E_{-\alpha}, E_\gamma \rangle = \delta_{\alpha\gamma}$, a relation which defines $E_{-\alpha}$. From this definition it follows that $E_{-\alpha}$ and $E_{-\beta}$ in eq. (4.8) are not highest weight. The above definition effectively make some of the basis elements imaginary (because the bilinear form is non-positive definite in general). However, the constraints are linear in the basis element, so a rescaling of the basis elements does not affect the end result. The $\hat{}$ in (4.8) acts in the following way on $\mathcal{D}_0 a$, $\mathcal{D}_0(Z) a \to -(D_Z + [\hat{W}(Z), \cdot\}) a$. For future reference we notice that the following closure relations hold

$$\begin{aligned} \sum_\alpha \hat{E}_\alpha \rangle \langle \hat{E}_{-\alpha} &= 1, \\ \sum_\alpha \hat{E}_{-\alpha} \rangle \langle E_\alpha &= 1. \end{aligned} \tag{4.9}$$

We want to show

$$\sum_\beta \int DY C_{\alpha\beta}(X,Y) C^{-1}_{\beta\gamma}(Y,Z) = \delta_{\alpha\gamma} \delta(X-Z). \tag{4.10}$$

In order to show this, i.e

$$-\sum_\beta \int DY \langle E_\alpha a U_0^{-1}(X) \hat{E}_\beta \rangle \delta(X-Y) \langle \hat{E}_{-\beta} \hat{U}_0(Y) A \hat{E}_{-\gamma} \rangle \delta(Y-Z) = \delta_{\alpha\gamma} \delta(X-Z), \tag{4.11}$$

we use the following property

$$\int DY\delta(X-Y)f(Y) = -f(X)\,. \tag{4.12}$$

We also use the fact that when moving U through DY it acquires a $\hat{\ }$. Equation (4.10) now follows using the closure relation, together with $\langle E_\alpha, \hat{E}_{-\gamma}\rangle = \langle E_{-\gamma}, E_\alpha\rangle = \delta_{\alpha\gamma}$. We can now write down the expression for the Dirac bracket between the components of the gauge-fixed current (the first term in the general expression for the Dirac bracket has been included)

$$\begin{aligned}
&\{\langle J(Z_1), T^A\rangle, \langle J(Z_2), T^B\rangle\}^* = -\langle \hat{T}^A \mathcal{D}_0(Z_1)\hat{T}^B\rangle\delta(Z_1-Z_2) - \\
&-\sum_{\alpha\beta}\int DX\int DY\{\langle J(Z_1), T^A\rangle, \chi_\alpha(X)\}C^{-1}_{\alpha\beta}(X,Y)\{\chi_\beta(Y), \langle J(Z_2), T^B\rangle\}\,.
\end{aligned} \tag{4.13}$$

Calculating the various factors using the definition of the second class constraints we can write the second term in (4.13) as

$$\begin{aligned}
&-\sum_{\alpha\beta}\int\!\!\int DX\,DY\langle \hat{T}^A\mathcal{D}_0(Z_1)a\hat{E}_\alpha\rangle\delta(Z_1-X)\cdot \\
&\qquad\cdot\langle \hat{E}_{-\alpha}\hat{U}_0(X)A\hat{E}_{-\beta}\rangle\delta(X-Y)\langle E_\beta a\mathcal{D}_0(Y)\hat{T}^B\rangle\delta(Y-Z_2).
\end{aligned} \tag{4.14}$$

Using the closure relations (4.9) together with (4.12), we get

$$-\langle \hat{T}^A\mathcal{D}_0(Z_1)aU_0(Z_1)\mathcal{D}_0(Z_1)\hat{T}^B\rangle\delta(Z_1-Z_2)\,. \tag{4.15}$$

We finally arrive at the following expression for the Dirac bracket in the lowest weight gauge

$$\{W_A(X), W_B(Y)\}^* = -\langle \hat{T}^A U_0(X)\mathcal{D}_0(X)\hat{T}^B\rangle\delta(X-Y)\,, \tag{4.16}$$

with $U = (1-\mathcal{D}_0 a)^{-1}$, and $W_A = \langle W, T^A\rangle$. We have thus constructed the bracket of the $\mathcal{W}$ algebra in the realization where the generators are linear in the current. It is a simple matter to (formally) transform to the language of SOPE's

$$W_A(Z_1)W_B(Z_2) = -\langle \hat{T}^A U_0(Z_1)\mathcal{D}_0(Z_1)\hat{T}^B\rangle\frac{\theta_{12}}{z_{12}}\,. \tag{4.17}$$

As an example we calculate the OPE between the superspin 1 generator W_1, the super energy momentum tensor $W_{\frac{3}{2}}$, and the other $\mathcal{W}$ algebra generators. The method used to derive the results (4.18) is simple: One starts by writing $W_A(Z_1)W_B(Z_2) \sim \langle \hat{T}^A\sum_n(a\mathcal{D}_0(Z_1))^n\mathcal{D}_0(Z_1)\hat{T}^B\rangle$. The next step is to let the operators a and $\mathcal{D}_0$ act on $\hat{T}^B$, and use the properties of the bilinear form (2.10) including the property (2.20). Since a decreases the grade and $\mathcal{D}_0$ never increases the grade only a finite number of terms in the sum above will contribute. The result is as expected

$$\begin{aligned}
W_1(Z_1)W_n(Z_2) &\sim \frac{\hat{c}}{2z_{12}^2}\delta_{s,1} + 2n\frac{\theta_{12}}{z_{12}}W_{\frac{2n+1}{2}}(Z_2)\,,\\
W_1(Z_1)W_{\frac{2n+1}{2}}(Z_2) &\sim -n\frac{\theta_{12}}{z_{12}^2}W_n(Z_2) - \frac{D_2W_n(Z_2)}{2z_{12}} - \frac{1}{2}\frac{\theta_{12}}{z_{12}}\partial_2 W_n(Z_2)\,,\\
W_{\frac{3}{2}}(Z_1)W_s(Z_2) &\sim \frac{\hat{c}}{4z_{12}^3}\delta_{s,\frac{3}{2}} + s\frac{\theta_{12}}{z_{12}^2}W_s(Z_2) + \frac{D_2W_s(Z_2)}{2z_{12}} + \frac{\theta_{12}}{z_{12}}\partial_2 W_s(Z_2)\,,
\end{aligned} \tag{4.18}$$

where $\hat{c} = 2d(d-1)$. The central charge in the Virasoro algebra is given by $c = \frac{3}{2}\hat{c} = 3d(d-1)$. When $d = 2$ only two generators survive, namely W_1 and $W_{\frac{3}{2}}$. W_1 contains one field of spin 1 and one of spin $\frac{3}{2}$; $W_{\frac{3}{2}}$ contains one field of spin $\frac{3}{2}$ and one of spin 2 (the energy momentum tensor). Together these generators span the well known $N = 2$ super conformal algebra [23, 24]. When $d = n$ we obtain instead the $N = 2$ supersymmetric W_n algebra. The general expression for the central charges in the algebra can also be given as

$$W_s(Z_1)W_{s'}(Z_2) \sim \frac{c_s}{z_{12}^{2s}}\delta_{s,s'} + \ldots \tag{4.19}$$

where

$$\begin{aligned} c_{\frac{2n+1}{2}} &= \frac{n}{2}c_n \\ c_n &= \frac{(-1)^{n+1}}{2^{n-1}}\frac{(n-1)!}{(2n-1)!!}\prod_{l=-n}^{l=n-1}(d+l)\,. \end{aligned} \tag{4.20}$$

We close this section by noting that it would be interesting if the $\mathcal{W}$ algebra bracket could be written in $N = 2$ superspace language, thereby making the $N = 2$ supersymmetry manifest, this would entail rewriting the formula (4.16) in $N = 2$ superspace.

5 The Miura Transformation

In this section we will describe a free field realization of the super $\mathcal{W}$ algebra introduced in the previous section. We will follow the treatment of the bosonic case [1]. The goal is to derive a closed expression for the $\mathcal{W}$ algebra generators expressed in terms of free superfields. We start by partially fixing the gauge in the following way

$$J \approx a^+ - D_+\Upsilon - W \tag{5.1}$$

where $\Upsilon \in \mathcal{G}_0$. The superfield $W \in \ker ad_{a^-}$ in (5.1) is in general different from the W used in section 4, however when we constrain Υ to zero by further gauge fixing, the two W's become equal. If we instead gauge-fix W in (5.1) to be zero, we obtain the so called diagonal gauge. This gauge is only accessible locally; but nevertheless it is extremely useful, since it gives rise to a free-field realization of the $\mathcal{W}$ algebra, since the components of Υ satisfy $D_+D_-\Upsilon_A = 0$, as well as free field (Dirac) superbrackets. The gauge fixing (5.1) imply that the remaining unfixed first class constraints are of the form

$$\varrho_\alpha = \langle J - a^+, ah_\alpha\rangle \approx 0\,, \tag{5.2}$$

where the h_α's span a basis in the subspace of grade zero elements. The $\mathcal{W}$ algebra is spanned by gauge invariant combinations of J's. Under a gauge transformation generated by the residual first class constraints (5.2) the $\mathcal{W}$ algebra generators are invariant i.e. $\delta\mathsf{W}[J] = \{\varrho(X), \mathsf{W}[J]\}^* \approx 0$. Here $\mathsf{W}[J]$ is an arbitrary $\mathcal{W}$ algebra generator. On the constraint surface we have $\mathsf{W}[J] = \mathsf{W}[D_+\Upsilon, W]$, since J is then of the form (5.1). For a constraint of the form $\varrho(X) = \langle J(X) - a^+, a\xi\rangle$ we get

$$\delta\mathsf{W} = \left[\int\{\varrho(X), W_A(Y)\}^* dY\frac{\delta}{\delta W_A(Y)} + \int\{\varrho(X), \partial_+\Upsilon_A(Y)\}^* dY\frac{\delta}{\delta D_+\Upsilon_A(Y)}\right]\mathsf{W} \approx 0\,. \tag{5.3}$$

Here we have used the definition $\frac{\delta}{\delta W_A(X)} W_B(Y) = \delta_{AB}\delta(X-Y)$, and similarly for $\frac{\delta}{\delta D_+\Upsilon_i(X)}$ The next step is to calculate the Dirac brackets appearing in (5.3). We start with $\{\varrho(X), D_+\Upsilon(Y)\}^*$. As in the bosonic case [1] it can be proven that only the first term in the general expression for the Dirac bracket contributes, hence

$$\{\varrho(X), \langle D_+\Upsilon(Y), h\rangle\}^* = \{\varrho(X), \langle D_+\Upsilon(Y), h\rangle\} = \langle \xi, h\rangle \delta(X-Y)\,. \tag{5.4}$$

The other Dirac bracket in (5.3) can be shown to equal

$$\begin{aligned}\{\varrho(X), \langle W(Y), T^A\rangle\}^* &= -\langle \xi a \mathcal{D}(X)\hat{T}^A\rangle\delta(X-Y) - \langle \xi a\mathcal{D}(X) a U(X)\mathcal{D}(X)\hat{T}^A\rangle\delta(X-Y)\\ &= -\langle \xi, V(X)\hat{T}^A\rangle\delta(X-Y)\,,\end{aligned} \tag{5.5}$$

where $U(X) = (1-\mathcal{D}(X)a)^{-1}$, $\mathcal{D}(X) = D_X + [D\Upsilon(X),\cdot\} + [W(X),\cdot\}$, and $V(X) = (1-a\mathcal{D}(X))^{-1}$. The proof of equation (5.5) follows closely the calculation performed in the bosonic case [1]. We can move the $\hat{}$ in (5.5) from T^A to $V(X)$, using the properties of the trace. Inserting the above results into (5.3) and performing the integration over Y, we obtain the equation

$$\left[\langle \xi, \frac{\delta}{\delta D_+\Upsilon}\rangle - \langle \xi, V(D_+\Upsilon, W)\frac{\delta}{\delta W}\rangle\right] \mathsf{W}[D_+\Upsilon, W] = 0\,, \tag{5.6}$$

where we have used the notation $\frac{\delta}{\delta W} = \sum_A T^A \frac{\delta}{\delta W_A}$. In this formula the T^A's are highest weight elements satisfying $W_A = \langle W, T^A\rangle$, which implies $\langle \eta, \frac{\delta}{\delta W(X)}\rangle W(Y) = \eta\delta(X-Y)$. The integrability condition of the variational equation (5.6) is satisfied as a consequence of the fact that the Dirac bracket satisfies the (super)Jacobi identity. It is possible to show that the solution to equation (5.6) is

$$\mathsf{W}[D_+\Upsilon, W] = \exp(-\int_0^1 ds \int dY \langle D_+\Upsilon(Y), V[sD_+\Upsilon(Y), W(Y)]\frac{\delta}{\delta W(Y)}\rangle)\mathsf{W}[W]\,. \tag{5.7}$$

It is possible, using the above formula, to derive expressions for the $\mathcal{W}$ algebra generators expressed in terms of the free superfield Υ, using $\mathsf{W}[D_+\Upsilon] = \mathsf{W}[D_+\Upsilon, W]|_{W=0}$, i.e. going to the diagonal gauge. As an example we give the first few generators. We get

$$\begin{aligned} W_1[D_+\Upsilon] &= \frac{1}{2}\langle D_+\Upsilon r D_+\Upsilon\rangle + \langle \frac{(K+2\nu)}{2} D_+^2\Upsilon\rangle\,,\\ W_{\frac{3}{2}}[D_+\Upsilon] &= -\frac{1}{2}\langle D_+\Upsilon D_+^2\Upsilon\rangle + \langle T^0 D_+^3\Upsilon\rangle\,,\end{aligned} \tag{5.8}$$

where the operator r has been defined in section 2. To obtain the first expression we made use of the fact that $K=r$ when acting on $\mathbf{Z}_2$ odd elements, and furthermore that $rA = Ar$ when acting on neutral elements. From (5.8) we see that the existence of r is crucial for the existence of the spin 1 generator. When $d=2$ the above expressions constitute a free field realization of the $N=2$ super conformal algebra. Using the same method as in the bosonic case it is possible to prove that $\mathsf{W}[D_+\Upsilon] = \mathsf{W}[-D_+\phi]$, i.e. that the $\mathcal{W}$ algebra generators are the same[8] when expressed in terms of the free fields

[8]The relative sign is a result of our conventions.

and the Toda fields. E.g. we see that the superspin $\frac{3}{2}$ generator above is precisely the (improved) energy momentum tensor of the Toda theory described in section 3. It is of course important to study the quantization of the general $\mathcal{W}$ algebra. We will return to this problem in a future publication.

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References

[1] N. Wyllard, *Generalized Toda Theories from WZNW Reduction*, hep-th/9603040.

[2] L. Brink and M.A. Vasiliev, *Generalized Toda Field Theories*, Nucl. Phys. **B457** (1995) 273.

[3] J. M. Evans and T. J. Hollowood, *Supersymmetric Toda Field Theories*, Nucl. Phys. **B352** (1991) 723.

[4] D.A. Leites, M.V. Saveliev and V.V. Sergonova, in: Group Theoretical Methods in Physics, Vol 1 , eds. M.A. Markov et al, VNU Science Press, 1986.

[5] M.A. Olshanetsky, *Supersymmetric Two-Dimensional Toda Lattice*, Commun. Math. Phys. **88** (1983) 63.

[6] F. Delduc, E. Ragoucy and P. Sorba, *Super-Toda Theories and W-algebras from Superspace Wess-Zumino-Witten Models*, Commun. Math. Phys. **146** (1992) 403.

[7] L.Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, *On Hamiltonian Reductions of the Wess-Zumino-Novikov-Witten Theories*, Phys. Rep. **222** (1992) 1.

[8] M.A. Vasiliev, *Quantization on Sphere and High-Spin Superalgebras*, JETP Lett. **50** (1989) 374.

[9] M.A. Vasiliev, *Higher Spin Algebras and Quantization on the Sphere and Hyperboloid*, Int. J. Mod. Phys. **A6** (1991) 1115.

[10] E. Bergshoeff, B. de Wit and M.A. Vasiliev, *The Structure of the Super-*$\mathrm{W}_\lambda(\infty)$ *Algebra*, Nucl. Phys **B366** (1990) 315.

[11] J. M. Evans, *Complex Toda Theories and Twisted Reality Conditions*, Nucl. Phys. **B390** (1993) 225.

[12] T. Inami and K.-I. Izawa, *Super-Toda Theory from WZNW Theories*, Phys. Lett **B255** (1991) 521.

[13] F. Delduc and M. Magro, *Gauge Invariant Formulation of* $N = 2$ *Toda and KdV Systems in Extended Superspace* hep-th/9512220.

[14] I. Tsutsui and L. Fehér, *Global Aspects of the WZNW Reduction to Toda Theories*, Prog. Theor. Phys. Supp. **118** (1995) 173.

[15] L. Brink and M. Vasiliev, unpublished.

[16] H. Nohara and K. Mohri, *Extended Superconformal Algebra From Super Toda Field Theory*, Nucl. Phys. **B349** (1991) 253.

[17] S. Komata, K. Mohri and H. Nohara, *Classical and Quantum Extended Superconformal Algebra*, Nucl. Phys. **B359** (1991) 168.

[18] L. Frappat, E. Ragoucy and P. Sorba, *W-algebras and Superalgebras from Constrained WZW Models: A Group Theoretical Classification*, Commun. Math. Phys. **157** (1993) 499.

[19] K. Huitu and D. Nemeschansky, *Supersymmetric Gelfand-Dickey Bracket*, Mod. Phys. Lett. **A6** (1991) 3179.

[20] J.M. Figueroa-O'Farrill and E. Ramos, *Classical $N = 2$ W-Superalgebras and Supersymmetric Gelfand-Dickey Brackets*, Nucl. Phys. **B368** (1992) 361.

[21] C. Hull and B. Spence, $N = 2$ *Current Algebra and Coset Models*, Phys. Lett. **B241** (1990) 357.

[22] C. Ahn, E. Ivanov and A. Sorin, $N = 2$ *Affine Superalgebras and Hamiltonian Reduction in $N = 2$ Superspace* hep-th/9508005.

[23] M. Ademollo et al, *Supersymmetric Strings and Colour Confinement*, Phys. Lett. **62B** (1976) 105.

[24] M. Ademollo et al, *Dual String with $U(1)$ Colour Symmetry*, Nucl. Phys. **B111** (1976) 77.